* Time-Variant Checking:

$$y(t-t_o) = x(t-t_o)\cos(\omega_o(t-t_o))$$

Now Let's delay $x(t)$ by $t_o \rightarrow x(t-t_o)$

$\therefore\ F[x(t-t_o)] = x(t-t_o)\cos(\omega_o t) \neq y(t-t_o)$

So the system is time-varying.

c) $y(t) = 3x(t-4)$

* Linearity Checking:

- Let $y_1(t) = 3x_1(t-4) = F[x_1]$, $y_2(t) = 3x_2(t-4) = F[x_2]$

$\therefore\ F[x_1] + F[x_2] = 3x_1(t-4) + 3x_2(t-4)$

- $x(t) = a_1 x_1(t) + a_2 x_2(t)$

$$\begin{aligned} F[a_1x_1(t) + a_2x_2(t)] &= 3[a_1x_1(t-4) + a_2x_2(t-4)] \\ &= 3a_1x_1(t-4) + 3a_2x_2(t-4)] \\ &= a_1F[x_1(t)] + a_2F[x_2(t)] \end{aligned}$$

So the system is Linear

* Time-Variant Checking:

$$y(t-t_o) = 3x(t-4-t_o)$$

Now let's delay $x(t)$ by $t_o \rightarrow x(t-t_o)$

$\therefore\ F[x(t-t_o)] = 3x(t-4-t_o) = y(t-t_o)$

So the system is time invariant.

As a result, this system is LTI system.

3. Response of LTI Systems:

[A] Impulse Response:

The impulse response $h(t)$ of LTI system is defined to be the response (output) of the system when the input is $\delta(t)$.

$$h(t) = F[\delta(t)]$$

where $h(t)$ is the impulse response of the system.

* **Causal System:** The system whose impulse response vanish for $t<0$:

$$h(t) = 0 \quad \text{for} \quad t<0$$

* This means that the causal system does not response before the excitation is applied.
* The response of LTI system to any input signal $x(t)$ in time domain is given by:

$$\boxed{y(t) = x(t) * h(t)} \quad \text{in time domain}$$

$$\therefore y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

B Frequency Response $H(f)$: (Transfer Function)

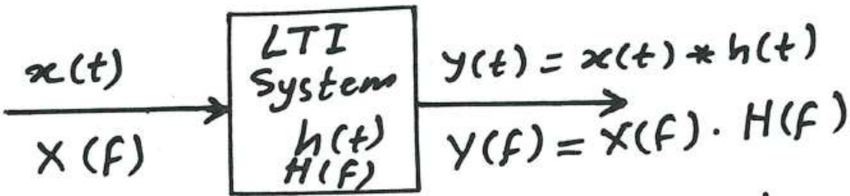
It is the Fourier transform of the impulse response.

$$h(t) \xleftrightarrow{F.T} H(f) \quad \text{or} \quad h(t) \xleftrightarrow{F.T} H(\omega)$$

$$y(t) = x(t) * h(t) \qquad \text{Take F.T for both sides}$$

$$Y(f) = X(f) \cdot H(f) \quad \{\text{Freq. Domain}\}$$

$$\therefore \boxed{H(f) = \frac{Y(f)}{X(f)}} \quad \text{OR} \quad H(\omega) = \frac{Y(\omega)}{X(\omega)}$$



- The transfer function $H(f)$ is a complex quantity,

$$H(f) = |H(f)|\, e^{j\theta_h(f)}$$

$$H^*(f) = H(-f)$$

and $|H(f)| = |H(-f)| \rightarrow$ even symmetry

$\theta_h(f) = -\theta_h(-f) \rightarrow$ odd symmetry

$$Y(f) = X(f)\, H(f)$$

so $|Y(f)| = |X(f)| \cdot |H(f)|$

$$\theta_y(f) = \theta_x(f) + \theta_h(f)$$

* Special Case:

Sinusoidal signal input does not need to be converted to frequency domain to deduce its response in LTI systems.

$$x(t) = A\cos(2\pi f_0 t + \theta_x)$$

$$H(f) = |H(f)|\, e^{j\theta_h(f)} = |H(f)| \angle \theta_h(f)$$

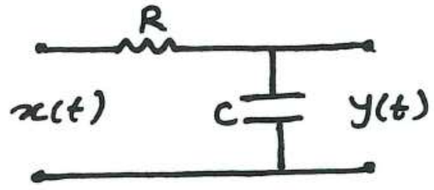
$\therefore \boxed{y(t) = |H(f_0)|\, A\cos\left(2\pi f_0 t + \theta_x + \theta_h(f_0)\right)}$ For sinusoidal only

Example: For the following RC network, find:

a) The frequency response $H(f)$.

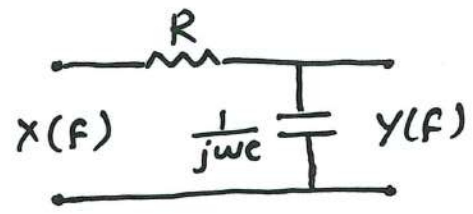
b) The impulse response $h(t)$.

c) Unit-step response.



Sol.

a) $$Y(f) = \frac{\frac{1}{j2\pi fC}}{R + \frac{1}{j2\pi fC}}\, X(f)$$



$$Y(f) = H(f)\, X(f)$$

$$\therefore H(f) = \frac{\frac{1}{j2\pi fC}}{R + \frac{1}{j2\pi fC}} \Rightarrow H(f) = \frac{1}{1 + j2\pi f RC}$$

OR, $H(f) = \dfrac{\frac{1}{RC}}{\frac{1}{RC} - j2\pi f}$

b) $h(t) = F.T^{-1}[H(f)]$

$$h(t) = F.T^{-1}\left[\frac{1}{RC}\,\frac{1}{\frac{1}{RC} + j2\pi f}\right] \equiv F.T^{-1}\left[A\,\frac{1}{a + j2\pi f}\right]$$

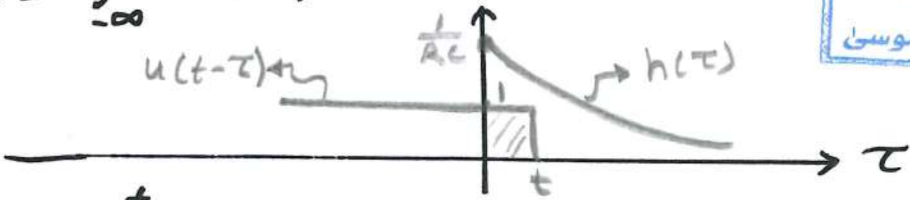
$$h(t) = \frac{1}{RC}\, e^{-t/RC}\, u(t)$$

c) Unit step response means $y(t)$ when $x(t) = u(t)$.

The time domain analysis is easier to be used in this problem.

$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau$$



$$y(t) = \int_0^t h(\tau) \times 1\, d\tau$$

$$y(t) = \int_0^t \frac{1}{RC} e^{\frac{-\tau}{RC}}\, d\tau = \frac{-RC}{RC}\, e^{-\frac{\tau}{RC}}\Big|_0^t$$

$$\therefore \quad y(t) = 1 - e^{-t/RC}$$

4. Distortionless Transmission:

The distortionless channel has an output signal that represents a delayed version of the input signal.

$$y(t) = k\, x(t - t_o)$$

where,

k - the gain constant. (attenuation if $k<1$)

t_o - the time delay.

$$y(t) = x(t) * k\,\delta(t - t_o) \equiv x(t) * h(t)$$

$$\therefore \quad Y(f) = X(f) \cdot k\, e^{-j(2\pi t_o) f}$$

$$Y(f) = X(f) \cdot H(f)$$

∴ The impulse response of the distortionless channel is:

$$\boxed{h(t) = k\,\delta(t - t_0)}$$

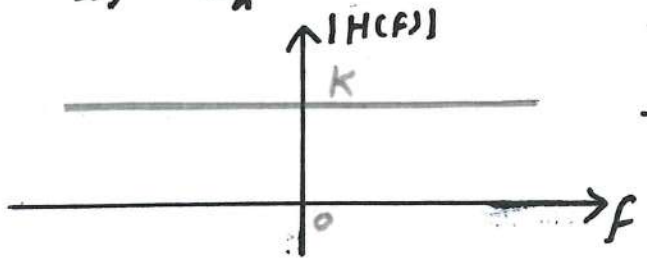
and its frequency response (transfer function) is:

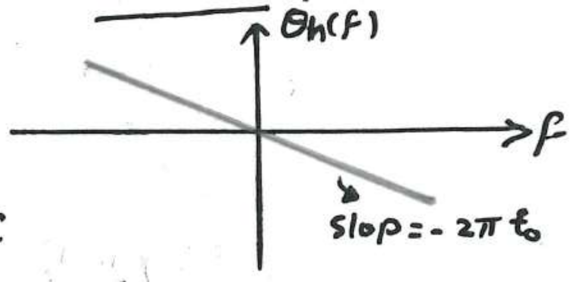
$$\boxed{H(f) = k\,e^{-j(2\pi t_0)f}} \equiv |H(f)|\,e^{j\theta_h(f)}$$

In other words, for distortionless channel:

1) $|H(f)| = k$ constant value

2) $\theta_h(f) = -(2\pi t_0)f$ linear phase.





A Linear Distortion in Linear Systems:

-Amplitude Distortion:

A distortion happens when $|H(f)|$ is not constant within the frequency range of interest. Therefore, the frequency components of the input signal are transmitted with different amount of attenuation.

- Phase Distortion:

A distortion happens when $\theta_h(f)$ is nonlinear within the frequency range of interest. Transmission has different delay for different frequency components

* The signals may undergo both distortions simultaneously.

Group and Phase Delay:

- Group Delay $T_g(f)$: It is the delay that a group of two or more frequency components undergo in passing through a linear system.

$$\boxed{T_g(f) = \frac{-1}{2\pi} \frac{d\,\theta_h(f)}{df}}$$

where $\theta_h(f)$ is the phase response of the system.

For distortionless system $\theta_h(f) = -2\pi f t_o$

$$\therefore \quad T_g(f) = \frac{-1}{2\pi} \frac{d}{df}(-2\pi f t_o)$$

$$\therefore \quad T_g(f) = \boxed{t_o}$$

- Phase Delay $T_p(f)$: It is the delay of single frequency component.

$$\boxed{T_p(f) = -\frac{\theta_h(f)}{2\pi f}}$$

For distortionless system

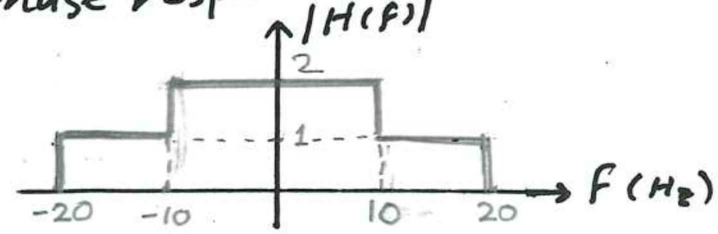
$$T_p(f) = -\frac{-2\pi f\, t_o}{2\pi f}$$

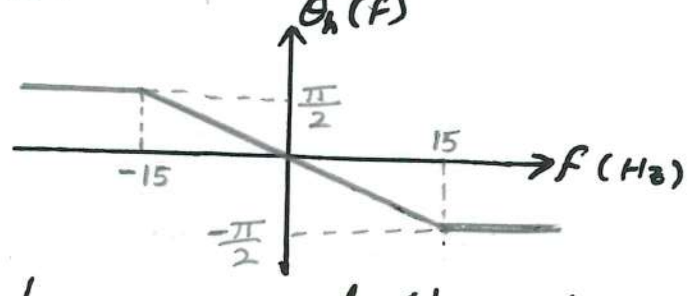
$$\therefore \quad T_p(f) = \boxed{t_o}$$

Thus, for distortionless channels

$$T_g = T_d = \text{constant.}$$

Example: Consider a system with amplitude and phase responses shown below:





Suppose the following signals are passed through the the above system.

a) $x_1(t) = \cos(10\pi t) + \cos(12\pi t)$

b) $x_2(t) = 3\cos(10\pi t) + 6\cos(26\pi t)$

c) $x_3(t) = \cos(26\pi t + 30°) + \sin(34\pi t)$

Find the response of each signal and the distortion type. Also determine and sketch the group delay and the phase delay.

Sol.

$$|H(f)| = \begin{cases} 1 & -20 < f < -10 \\ 2 & -10 < f < 10 \\ 1 & 10 < f < 20 \end{cases} \qquad \theta_h(f) = \begin{cases} \frac{\pi}{2} & f < -15 \\ -\frac{\pi}{30} f & -15 < f < 15 \\ \frac{\pi}{2} & f > 15 \end{cases}$$

Remember that $y(t) = |H(f_0)|\, A\cos(2\pi f_0 t + \theta + \theta_h(f_0))$

a) $x_1(t) = \cos(2\pi(5)t + 0°) + \cos(2\pi(6)t + 0°)$

$f_1 = 5\,Hz \qquad f_2 = 6\,Hz$

$\therefore\ y_1(t) = \underline{\underline{2}} \times \cos\left(2\pi(5)t - \frac{\pi}{30}(5)\right) + \underline{\underline{2}} \times \cos\left(2\pi(6)t - \frac{\pi}{30}(6)\right)$

$y_1(t) = 2\cos\left(2\pi(5)t - \frac{\pi}{6}\right) + 2\cos\left(2\pi(6)t - \frac{\pi}{5}\right)$

Since both components multiplied by the same gain $(k=2)$, and the phase are linear through the both frequencies, then this signal is passed <u>without any distortion</u> (distortionless transmission).

b) $x_2(t) = 3\cos(2\pi(5)t + 0°) + 6\cos(2\pi(13)t + 0°)$

$f_1 = 5\,Hz \qquad f_2 = 13\,Hz$

$y_2(t) = \underline{\underline{2}} \times 3\cos\left(2\pi(5)t - \frac{\pi}{30}\times 5\right) + \underline{\underline{1}} \times 6\cos\left(2\pi(13)t - \frac{\pi}{30}\times 13\right)$

$y_2(t) = 6\cos\left(2\pi(5)t - \frac{\pi}{30}\times 5\right) + 6\cos\left(2\pi(13)t - \frac{\pi}{30}\times 13\right)$

$y_2(t) = 6\cos\left(2\pi(5)t - \frac{\pi}{6}\right) + 6\cos\left(2\pi(13)t - \frac{13}{30}\pi\right)$

The 1st component is multiplied by gain = 2
The 2nd component is multiplied by gain = 1
but both components locates at the same linear region of the phase response, so
$x_2(t)$ undergoes amplitude distortion only.

c) $x_3(t) = \cos(2\pi(13)t + 30°) + \cos(2\pi(17)t - 90°)$
$f_1 = 13\ Hz$ $\qquad f_2 = 17\ Hz$

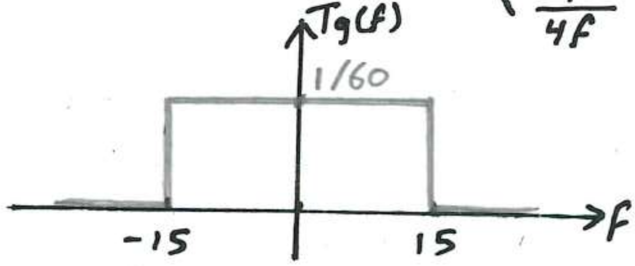
$$y_3(t) = \underline{1} * \cos(2\pi(13)t + 30° - \frac{\pi}{30} \times 13) + \underline{1} * \cos(2\pi(17)t - 90° - \frac{\pi}{2})$$

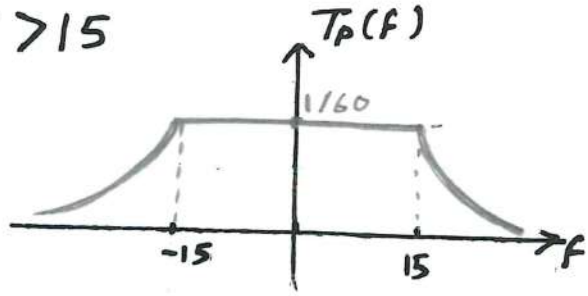
$$y_3(t) = \cos(2\pi(13)t - 48°) + \cos(2\pi(17)t - 180°)$$

Both components are multiplied by gain = 1
but they do not located within the same linear phase response
$x_3(t)$ undergoes phase distortion only.

$$T_g(f) = \frac{-1}{2\pi} \cdot \frac{d\,\theta_h(f)}{df} = \begin{cases} 0 & f < -15 \\ \frac{1}{60} & -15 < f < 15 \\ 0 & f > 15 \end{cases}$$

$$T_d(f) = \frac{\theta_h(f)}{-2\pi f} = \begin{cases} \frac{-1}{4f} & f < -15 \\ \frac{1}{60} & -15 < f < 15 \\ \frac{1}{4f} & f > 15 \end{cases}$$





B Nonlinear Distortion:

This kind of distortion is introduced by nonlinear systems. Suppose an input signal $x(t)$ given by:

$$x(t) = A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t)$$

is entered to a nonlinear system whose input-output characteristic is given by:

$$y(t) = a_1\, x(t) + a_2\, x^2(t)$$

where a_1 & a_2 are constants. Therefore,

$$y(t) = a_1\left[A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t)\right] + a_2\left[A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t)\right]^2$$

$$y(t) = a_1\left[A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t)\right] +$$

$$+ a_2\left[A_1^2 \cos^2(2\pi f_1 t) + 2A_1A_2 \cos(2\pi f_1) \cos(2\pi f_2 t) + A_2^2 \cos^2(2\pi f_2 t)\right]$$

but $\cos^2\theta = \frac{1}{2}\left[1 + \cos(2\theta)\right]$

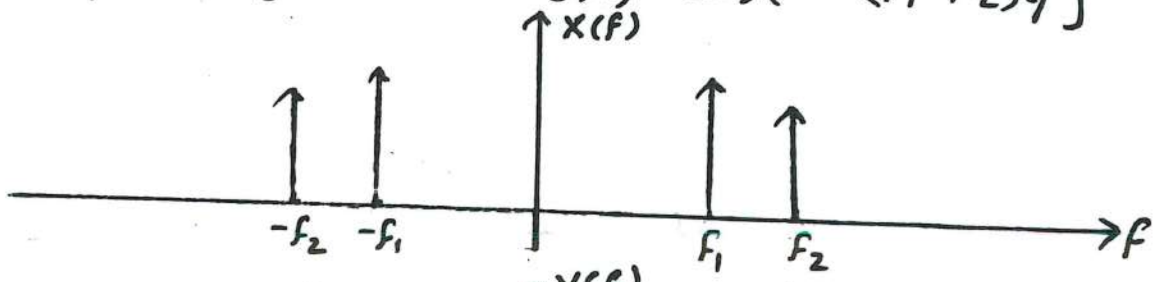
$$\cos\theta_1 \cos\theta_2 = \frac{1}{2}\left[\cos(\theta_1 - \theta_2) + \cos(\theta_1 + \theta_2)\right]$$

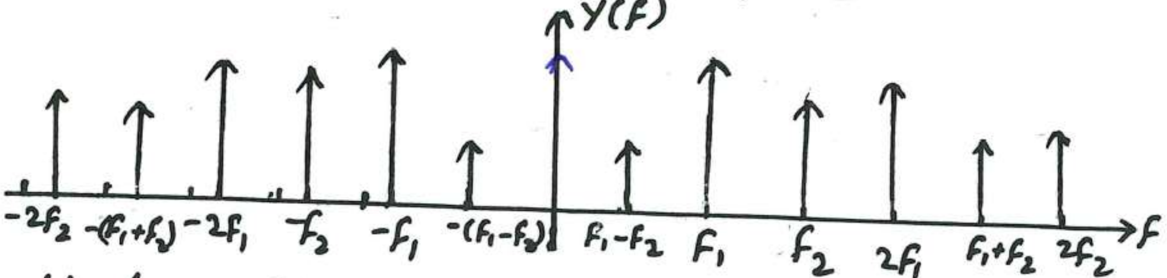
$$\therefore\ y(t) = a_1\left[A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t)\right] +$$

$$+ \frac{1}{2} a_2 (A_1^2 + A_2^2) +$$

$$+ \frac{1}{2} a_2\left[A_1^2 \cos(2\pi (2f_1) t) + A_2^2 \cos(2\pi (2f_2) t)\right] +$$

$$+ a_2 A_1 A_2\left[\cos(2\pi (f_1 + f_2) t) + \cos(2\pi (f_1 - f_2) t)\right]$$





<u>Note that</u> nonlinear systems has produced frequencies in the output other than the input freqencies.

5. Filters:

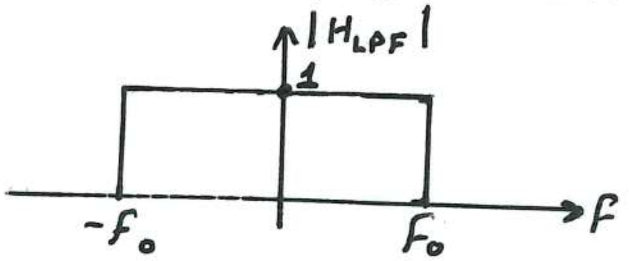
The filter is a system that passes some frequency components and supresses others.

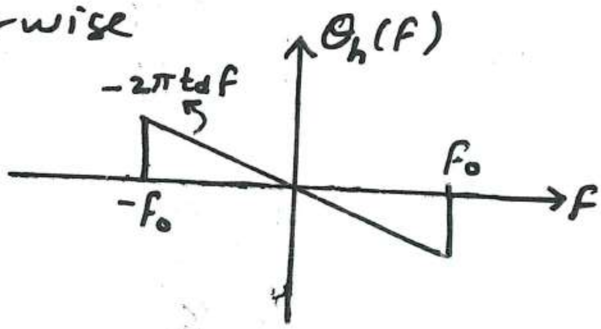
A Ideal Filters:

Filters have the characteristics of distortionless transmission over specified frequency band and zero response over all other bands.

- Ideal Low-Pass Filter (LPF):

$$H_{LPF}(f) = \begin{cases} e^{-j2\pi t_d f} & |f| \leq f_o \\ 0 & \text{otherwise} \end{cases}$$





It can be re-written as:

$$\boxed{H_{LPF}(f) = \Pi\left(\frac{f}{2f_o}\right) e^{-j2\pi t_d f}}$$

Ideal LPF can pecfectly pass frequencies below f_o and perfectly eliminate frequencies above f_o.

$$h_{LPF}(t) = F.T^{-1}[H_{LPF}(f)]$$

$$h_{LPF}(t) = 2f_o \, \text{Sinc}[2f_o(t - t_d)]$$